

An inverse problem with data on the part of the boundary

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Abstract

Let $u_t = \nabla^2 u - q(x)u := Lu$ in $D \times [0, \infty)$, where $D \subset R^3$ is a bounded domain with a smooth connected boundary S , and $q(x) \in L^2(S)$ is a real-valued function with compact support in D . Assume that $u(x, 0) = 0$, $u = 0$ on $S_1 \subset S$, $u = a(s, t)$ on $S_2 = S \setminus S_1$, where $a(s, t) = 0$ for $t > T$, $a(s, t) \not\equiv 0$, $a \in C([0, T]; H^{3/2}(S_2))$ is arbitrary.

Given the extra data $u_N|_{S_2} = b(s, t)$, for each $a \in C([0, T]; H^{3/2}(S_2))$, where N is the outer normal to S , one can find $q(x)$ uniquely. A similar result is obtained for the heat equation $u_t = \mathcal{L}u := \nabla \cdot (a \nabla u)$.

These results are based on new versions of Property C.

1 Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth connected boundary S , $S = S_1 \cup S_2$, S_1 is an open subset in S , and S_2 is the complement of S_1 in S . Consider the problem

$$u_t = \nabla^2 u - q(x)u, \quad (x, t) \in D \times [0, \infty), \quad (1)$$

$$u(x, 0) = 0, \quad u|_{S_1} = 0, \quad u|_{S_2} = a(s, t), \quad (2)$$

where $q(x) \in L^2(D)$ is a real-valued function, $a(s, t) \in C([0, \infty), H^{3/2}(S_2))$, the function a is an arbitrary real-valued function in the above set such that $a \not\equiv 0$, $a = 0$ for $t > T$, H^ℓ is the Sobolev space.

MSC: 35K20, 35R30; PACS 02.30.Jr

Key words: Property C, parabolic equations, inverse problems

For each a in the above set let the extra data be given:

$$u_N|_{S_2} = b(s, t), \quad (3)$$

where N is the outer normal to the boundary S .

Do the data $\{a(s, t), b(s, t)\} \forall a(s, t) \in C([0, \infty), H^{3/2}(S_2))$, $a \not\equiv 0$, $a = 0$ for $t > T$, determine $q(x)$ uniquely?

Our main result is a positive answer to this question.

Theorem 1. *The data $\{a(s, t), b(s, t)\}_{\forall t \geq 0, \forall s \in S_2}$, given for all a with the above properties, determine a compactly supported in D real-valued function $q(x) \in L^2(D)$ uniquely.*

Actually we prove a slightly stronger result: the data for $0 \leq t \leq T + \varepsilon$ determine q uniquely, where $\varepsilon > 0$ is an arbitrary small number.

Note that the set S_2 can be arbitrary small.

Theorem 1 is a multidimensional generalization of the author's result from [1]. Let

$$L_j v - \lambda v := \nabla^2 v - q_j(x)v - \lambda v, \quad j = 1, 2,$$

and

$$N_j = \{v : L_j v - \lambda v = 0 \text{ in } D, v|_{S_1} = 0\}.$$

The main tool in the proof of Theorem 1 is a new version of property C. Originally this property was introduced by the author in 1986 for the products of solutions to homogeneous linear partial differential equations in the case when these solutions did not satisfy any boundary conditions (see [3]).

Theorem 2. *The set $\{v_1, v_2\}$ for all $v_1 \in N_1$ and all $v_2 \in N_2$ is complete (total) in $L^2(D_1)$, where $D_1 \subset D$ is a strictly inner subdomain of D , i.e., if $p \in L^2(D_1)$, $p = 0$ in $D \setminus D_1$, and $\int_{D_1} p v_1 v_2 dx = 0 \quad \forall v_1 \in N_1, \forall v_2 \in N_2$, then $p = 0$.*

In Section 2 Theorems 1 and 2 are proved. In Section 3 the results are generalized to the boundary-value problems for the equation $u_t = \nabla \cdot (a(x) \nabla u)$.

2 Proofs

Proof of Theorem 1. Let

$$v = \int_0^\infty e^{-\lambda t} u dt \quad \text{and} \quad A(s, \lambda) = \int_0^\infty e^{-\lambda t} a(s, t) dt.$$

Taking the Laplace transform of the relations (1)-(2), we obtain:

$$Lv - \lambda v = 0, \quad Lv = \nabla^2 v - q(x)v, \quad v|_{S_1} = 0, \quad v|_{S_2} = A(s, \lambda), \quad (4)$$

and

$$v_N = B(s, \lambda) = \int_0^\infty e^{-\lambda t} b(s, t) dt.$$

Assume that there are q_1 and q_2 , compactly supported in D , which generate the same data. Let

$$L_j v_j = \nabla^2 v_j - q_j v_j, \quad j = 1, 2,$$

and

$$w = v_1 - v_2.$$

Then

$$L_1 w - \lambda w = p v_2, \quad p = q_1 - q_2, \quad (5)$$

and for any

$$\psi \in N(L_1 - \lambda) := \{\psi : (L_1 - \lambda)\psi = 0\}, \quad \psi|_{S_1} = 0,$$

one gets

$$\int_D p v_2 \psi dx = \int_{S_2} (w_N \psi - \psi_N w) ds = 0, \quad (6)$$

because

$$w = \psi = 0 \quad \text{on} \quad S_1$$

and

$$w = w_N = 0 \quad \text{on} \quad S_2$$

by our assumptions. By Theorem 2, relation (6) implies $p = 0$.

Theorem 1 is proved. \square

Proof of Theorem 2. It is proved in [2] that the set $\{v_1 v_2\}$ for all

$$v_j \in M_j := \{v_j : (L_j - \lambda)v_j = 0 \text{ in } D\}$$

is total in $L^2(D)$. Therefore it is sufficient to prove that N_j is dense in M_j in $L^2(D_1)$ -norm, where D_1 is a strictly inner subdomain of D out of which both q_1 and q_2 vanish.

Let us take $j = 1$. The proof for $j = 2$ is the same. Assume the contrary. Then there is a $\psi \in M_1$ such that

$$0 = \int_{D_1} \psi v dx \quad \forall v \in N_1. \quad (7)$$

Let $G(x, y)$ solve the problem

$$(L_1 - \lambda)G = -\delta(x - y) \text{ in } \mathbb{R}^3, \quad \lim_{|x-y| \rightarrow \infty} G(x, y) = 0, \quad G|_{S_1} = 0. \quad (8)$$

Since $G(x, y) \in N_1 \quad \forall y \in D'_1 = \mathbb{R}^3 \setminus D_1$, equation (7) implies

$$0 = \int_{D_1} \psi(x) G(x, y) dx := h(y), \quad \forall y \in D'_1. \quad (9)$$

We have $h(y) \in H_{loc}^2(\mathbb{R}^3 \setminus S_1)$, and h solves the elliptic equation

$$(\nabla^2 - \lambda)h = 0 \quad \text{in } D'_1,$$

because $q_1 = q_2 = 0$ in D'_1 . Therefore, by the uniqueness of the solution to the Cauchy problem for elliptic equations, one gets from (9) the following relations:

$$h = h_N = 0 \text{ on } \partial D_1, \quad (10)$$

and, because of (8), one gets

$$(L_1 - \lambda)h = -\psi \text{ in } D_1. \quad (11)$$

Multiply (11) by $\bar{\psi}$, integrate over D_1 , use (10), and get

$$-\int_{D_1} |\psi|^2 dx = \int_{\partial D_1} (h_N \bar{\psi} - h \bar{\psi}_N) ds = 0. \quad (12)$$

Thus $\psi = 0$ in D_1 and, therefore, $\psi = 0$ in D , because ψ solves a homogeneous linear elliptic equation for which the uniqueness of the solution to the Cauchy problem holds. Theorem 2 is proved. \square

3 Generalizations

Consider now the problem

$$u_t = \nabla \cdot (a(x) \nabla u) := \mathcal{L}u, \quad (x, t) \in D \times [0, \infty), \quad (13)$$

$$u(x, 0) = 0, \quad u|_{S_1} = 0, \quad u|_{S_2} = h(s, t), \quad (14)$$

and the extra data are

$$au_N|_{S_2} = z(s, t) \quad \forall s \in S_2, \forall t > 0. \quad (15)$$

We assume that

$$0 < a_0 \leq a(x) \leq a_1,$$

where a_0 and a_1 are constants, $a \in H^3(D)$, and prove the following theorem.

Theorem 3. *Under the above assumptions, the data $\{h(s, t), z(s, t)\}_{\forall s \in S_2, \forall t > 0}$ determine $a(x)$ uniquely.*

Proof. Taking the Laplace transform of relations (13)-(15), we reduce the problem to

$$(\mathcal{L} - \lambda)v = 0 \text{ in } D, \quad v|_{S_1} = 0, \quad v|_{S_2} = H(s, \lambda), \quad av_N|_{S_2} = Z(s, \lambda), \quad (16)$$

where, e.g.,

$$H(s, \lambda) = \int_0^\infty e^{-\lambda t} h(s, t) dt.$$

Assuming that $a_j, j = 1, 2$, generate the same data $\{H(s, \lambda), Z(s, \lambda)\}_{\forall s \in S_2, \forall \lambda > 0}$, one derives for

$$w = v_1 - v_2$$

the problem

$$(\mathcal{L}_1 - \lambda)w = \nabla \cdot (p \nabla v_2), \quad p := a_2 - a_1, \quad w|_S = 0, \quad a_1 v_{1N}|_{S_2} = a_2 v_{2N}|_{S_2}. \quad (17)$$

Multiply (17) by an arbitrary element of \mathcal{N}_1 , where

$$\mathcal{N}_j = \{\varphi : (\mathcal{L}_j - \lambda)\varphi = 0 \text{ in } D, \quad \varphi|_{S_1} = 0\}, \quad j = 1, 2,$$

integrate by parts, and get

$$-\int_D p \nabla v_2 \nabla \varphi \, dx + \int_S p v_{2N} \varphi \, ds = \int_S (a_1 w_N \varphi - a_1 \varphi_N w) \, ds. \quad (18)$$

Using boundary conditions (17), one gets

$$\int_D p \nabla \varphi \nabla v_2 \, dx = 0 \quad \forall \varphi \in \mathcal{N}_1 \quad \forall v_2 \in \mathcal{N}_2. \quad (19)$$

To complete the proof of Theorem 3, we use the following new version of Property C.

Lemma 1. *The set $\{\nabla \varphi \cdot \nabla v_2\}_{\forall \varphi \in \mathcal{N}_1, \forall v_2 \in \mathcal{N}_2}$ is complete in $L^2(D)$ for all sufficiently large $\lambda > 0$.*

We prove this lemma below, but first let us explain the claim made in the Introduction:

Claim: The results remain valid if the data are given not for all $t \geq 0$ but for $t \in [0, T + \varepsilon]$, where $\varepsilon > 0$ is arbitrarily small.

This claim follows from the analyticity with respect to time of the solution $u(x, t)$ to problems (1)-(2) and (13)-(14) in a neighborhood of the ray (T, ∞) for an arbitrary small $\varepsilon > 0$. This analyticity holds if $a(s, t)$ and $h(s, t)$ vanish in the region $t > T$.

Proof of Lemma 1. It was proved in [2, pp.78-80], that the set $\{\nabla \psi_1 \cdot \nabla \psi_2\}_{\forall \psi_j \in \mathcal{M}_j}$ is complete in $L^2(D)$, where

$$\mathcal{M}_j = \{\psi : (\mathcal{L}_j - \lambda)\psi_j = 0\},$$

$$\mathcal{L}_j \psi = \nabla \cdot (a_j \nabla \psi) - \lambda \psi,$$

where $\lambda = \text{const} > 0$, $0 < c \leq a_j(x) \leq C$, $a_j(x) \in H^3(D)$, $j = 1, 2$, and $H^\ell(D)$ is the usual Sobolev space.

Lemma 1 will be proved if we prove that any $\psi_j \in \mathcal{M}_j$ can be approximated with an arbitrary small error in the norm, generated by the bilinear form

$$\int_D (a_j \nabla \psi_j \nabla \varphi + \lambda \psi_j \varphi) dx$$

by an element $v_j \in \mathcal{N}_j$. The above norm is equivalent to the norm of $H^1(D)$ due to the assumption $0 < c \leq a_j(x) \leq C$, $j = 1, 2$.

Assuming that such an approximation is not possible, we can find a $\psi_j \in \mathcal{M}_j$ such that

$$0 = \int_D [a_j(x) \nabla \psi_j \nabla G_j(x, y) + \lambda \psi_j G(x, y)] dx \quad \forall y \in D', \quad (20)$$

because $G_j(x, y) \in \mathcal{N}_j$ for any $y \in D'$. Integrating by parts in (20) and using the relation $G_j \in \mathcal{N}_j$, one gets

$$0 = \int_S a_j \psi_j G_{jN}(s, y) ds := h(y), \quad \forall y \in D'. \quad (21)$$

Denote

$$a_j \psi_j = \varphi, \quad G_j = G, \quad \mathcal{L}_j = \mathcal{L}.$$

Since

$$(\mathcal{L} - \lambda)G = -\delta(x - y) \text{ in } \mathbb{R}^3 \setminus S_1 \text{ and } G|_{S_1} = 0, \quad (22)$$

one can derive the relation

$$\lim_{y \rightarrow p \in S_1} G_N(s, y) = \frac{\delta(s - p)}{a(s)}, \quad (23)$$

where $\delta(s - p)$ is the delta-function on S_1 . We prove this relation later, but assuming that (23) holds we conclude from (21) that

$$a_j \psi_j := \varphi = 0 \quad \text{on} \quad S_1. \quad (24)$$

Consequently

$$h(y) = \int_{S_2} \varphi(s) G_N(s, y) ds = 0 \quad \forall y \in D'. \quad (25)$$

It follows from (25) that

$$(\mathcal{L} - \lambda)h = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus S_2, \quad h = 0 \text{ in } D'. \quad (26)$$

This and the uniqueness of the solution of the Cauchy problem for elliptic equation (25), imply

$$h = 0 \text{ in } \mathbb{R}^3 \setminus S_2. \quad (27)$$

From (27) and the jump relation for the double-layer potential (25), we conclude that

$$\varphi = 0 \text{ on } S_2. \quad (28)$$

From (24) and (28) it follows that $\varphi = 0$ on S . Therefore $\psi_j = 0$ on S because $a_j > 0$. Thus, $\psi_j \in \mathcal{M}_j$ and $\psi_j = 0$ on S . This implies $\psi_j = 0$ in D for all sufficiently large $\lambda > 0$.

Thus, Lemma 1 is proved if (23) is established.

Let us prove (23). Denote by g the (unique) solution to the problem

$$(\mathcal{L} - \lambda)g = -\delta(x - y) \text{ in } \mathbb{R}^3, \quad \lim_{|x-y| \rightarrow \infty} g(x, y) = 0. \quad (29)$$

Using Green's formula, we obtain from (22) and (29) the relation

$$G(x, y) = g(x, y) - \int_{S_1} g(x, s)a(s)G_N(s, y)ds, \quad x, y \in S'_1 := \mathbb{R}^3 \setminus S_1. \quad (30)$$

Taking $y \rightarrow p \in S_1$ and using the boundary condition (22), we derive:

$$g(x, p) = \lim_{y \rightarrow p} \int_{S_1} g(x, s)a(s)G_N(s, y)ds \quad \forall x \in S'_1. \quad (31)$$

Since the set $\{g(x, t)\}_{\forall x \in S'_1}$ is dense in $L^2(S_1)$, formula (31) implies (23).

Let us finally check the claim that the set $\{g(x, p)\}_{\forall x \in S'_1}$ is dense in $L^2(S_1)$. If it is not dense, then there is an $f \in L^2(S_1)$ such that

$$0 = \int_{S_1} f(p)g(x, p)dp := W(x), \quad \forall x \in S'_1. \quad (32)$$

The integral in (32) is a simple-layer potential $W(x)$ the density $f(p)$ of which must vanish because of the jump formula for the normal derivatives of W across S_1 . Thus the claim is proved. The proof of Lemma 1 is complete. Therefore Theorem 3 is proved. \square

Remark. If the conclusion of Lemma 1 remains valid for the set $\{\nabla\varphi \cdot \nabla v_2\}_{\forall \varphi \in \mathcal{N}_1, u_2 \in \mathcal{N}_2}$, where u_2 is a single element of \mathcal{N}_2 , possibly chosen in a special way, then the conclusion of Theorem 1 will be established for the data which is a single pair of data $\{a(s, t), b(s, t)\}_{\forall t \geq 0, \forall s \in S_2}$, where $a(s, t) \not\equiv 0$ is some function.

References

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